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Steinberg–Leibniz algebras and superalgebras

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Abstract

As a universal central extension of the special linear Lie algebra $\mathfrak{sl}(n, A)$ over a unital associative algebra A, the Steinberg algebras $\mathfrak{st}(n, A)$ and $\mathfrak{stl}(n, A)$ were studied in several papers. In this paper, we mainly study the Steinberg–Leibniz algebra $\mathfrak{stl}(n, D)$ defined over a dialgebra D. We prove that it is the universal central extension of the special linear Leibniz algebra $\mathfrak{sl}(n, D)$ with kernel $HHS_1(D)$, the quotient of the first Hochschild homology group $HH_1(D)$ of the dialgebra D by the ideal generated by $a \otimes (b \dashv c) - a \otimes (b \vdash c)$ for all $a, b, c \in D$. We also obtain a similar theorem for the Steinberg–Leibniz superalgebra $\mathfrak{stl}(m, n, D)$. This research plays a key role in studying the Leibniz algebras (superalgebras) graded by finite root systems and is also connected with 'Leibniz K-theory.'

Keywords: Steinberg–Leibniz algebras; Dialgebras; Central extension

1. Introduction

In [20], J.-L. Loday introduces a non-antisymmetric version of Lie algebras, whose bracket satisfies the Leibniz relation (see (2.7)) and therefore is called *Leibniz algebra*. The Leibniz relation, combined with antisymmetry, is a variation of the Jacobi identity. Hence Lie algebras are anti-symmetric Leibniz algebras. In [23], Loday also introduces

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an 'associative' version of Leibniz algebras, called *associative dialgebras*, equipped with two binary operations, $\vdash$ and $\dashv$, which satisfy the five relations (see the axiom (Ass) in Section 2). These identities are all variations of the associative law, so associative algebras are associative dialgebras for which the two products coincide. The peculiar point is that the bracket $[a, b] =: a \dashv b - b \vdash a$ defines a Leibniz algebra which is not antisymmetric, unless the left and right products coincide. Hence associative dialgebras yield a commutative diagram of categories and functors

$$\begin{array}{ccc} \mathbf{Dias} & \xrightarrow{-} & \mathbf{Leib} \\ \downarrow & & \downarrow \\ \mathbf{Ass} & \xrightarrow{-} & \mathbf{Lie}. \end{array}$$

By definition, the *Steinberg–Lie algebra* $\mathfrak{st}(n, A)$ over a K-algebra A is a Lie algebra generated by symbols $u_{ij}(a)$, $1 \leqslant i \neq j \leqslant n$, $a \in A$, subject to the relations

$$u_{ij}(k_1a + k_2b) = k_1u_{ij}(a) + k_2u_{ij}(b), \quad \text{for } a, b \in A,\ k_1, k_2 \in K; \tag{1.1}$$

$$[u_{ij}(a), u_{kl}(b)] = 0, \quad \text{if } i \neq l \text{ and } j \neq k; \tag{1.2}$$

$$[u_{ij}(a), u_{kl}(b)] = u_{il}(ab), \quad \text{if } i \neq l \text{ and } j = k. \tag{1.3}$$

It is clear that the relation (3) makes sense only if $n \geqslant 3$.

From [7] we see that the map $\eta : a \to u_{ij}(a)$ is one-to-one if and only if A is an associative algebra for $n \geqslant 4$ and A is an alternative algebra for $n = 3$.

Central extensions play an important role in the theory of Lie algebras, and it is therefore not surprising that there are many researches on central extensions of various classes of Lie algebras. The universal central extensions of $\mathfrak{sl}(n, A)$ over an associative algebra A ($n \geqslant 4$) or alternative algebra A ($n = 3$) in the categories of Lie algebras and Leibniz algebras were given in [2,10,12,25]. Moreover, S. Berman and R.V. Moody [4] proved that any Lie algebra graded by finite root system of A_l ($l \geqslant 2$) is centrally isogenous with the Steinberg Lie algebra $\mathfrak{st}(l + 1, A)$ for some unital associative algebra A if $l \geqslant 3$ and for some unital alternative algebra A if $l = 2$ ($\operatorname{char} K = 0$).

It is well known that the universal central extension of the matrix Lie algebra $\mathfrak{sl}(n, A)$ ($n \geqslant 3$) over a unital associative algebra A is the Steinberg algebra $\mathfrak{st}(n, A)$ [13]. But the universal central extension of $\mathfrak{sl}(n, A)$ ($n \geqslant 3$) in the category of Leibniz algebras is the Leibniz algebra $\mathfrak{stl}(n, A)$, which has the same generators and relations as $\mathfrak{st}(n, A)$ [12,25].

Motivated by [13,25], we consider the algebra $\mathfrak{sl}(n, D)$ with D a unital associative dialgebra with bracket:

$$\bigl[E_{ij}(a), E_{kl}(b)\bigr] = \delta_{jk}E_{il}(a \dashv b) - \delta_{il}E_{kj}(b \vdash a)$$

for all $a, b \in D$ and $1 \leqslant i \neq j$, $k \neq l \leqslant n$, where $E_{ij}(a)$ is the $n \times n$ matrix with coefficient a on (i, j)-th position and 0 in all others. Clearly with this bracket $\mathfrak{sl}(n, D)$ is not a Lie algebra in general but a Leibniz algebra. So it is interesting to consider the universal central extension of $\mathfrak{sl}(n, D)$. With this considerations we construct the *Steinberg–Leibniz* *algebra* $\mathfrak{stl}(n, D)$ for a unital dialgebra (not necessarily associative) D. It is a Leibniz algebra generated by symbols $v_{ij}(a)$, $1 \leqslant i \neq j \leqslant n$, $a \in D$, subject to the relations (3.4)–(3.7) in Section 3. As in [7] we obtain a sufficient and necessary condition for which the map $\eta : a \to v_{ij}(a)$ is one-to-one, and then introduce the concept of 'alternative dialgebra,' which is just generalization of alternative algebras. Moreover, when D is associative, we prove that $\mathfrak{stl}(n, D)$ $(n \geqslant 3)$ is also the universal central extension of $\mathfrak{sl}(n, D)$ with kernel $HHS_1(D)$ (see below). This research plays a key role in studying Leibniz algebras graded by finite root systems [15] as in the Lie algebra case [3,4] is also connected with 'Leibniz K-theory' [24].

In Section 2, we recall some notions of Leibniz algebras and dialgebras and give the definition of alternative dialgebras according to the definition of alternative algebras. In Section 3 we give the definition and some properties of $\mathfrak{st}(n, D)$. In Sections 4 and 5 we mainly prove that $\mathfrak{stl}(n, D)$ $(n \geqslant 3)$ is the universal central extension of $\mathfrak{sl}(n, D)$ with kernel $HHS_1(D)$ in the category of Leibniz algebras for a unital associative dialgebra D. The proof of the main theorem follows the same pattern as in [2] and [13]. In Section 6, we extend the results in Sections 4 and 5 to the Steinberg–Leibniz superalgebra $\mathfrak{stl}(m, n, D)$. Throughout this paper, K denotes a field. D denotes a unital dialgebra over K.

2. Dialgebras and Leibniz algebras

We recall some notions of associative dialgebras and Leibniz algebras and their (co)homology (see [5,8,9,11,12,14,20–23,25]). We also give the definition of alternative dialgebras according to the definition of alternative algebras.

2.1. Dialgebras

Definition 2.1 [23]. A dialgebra D over K is a K-vector space D with two linear operations $\dashv, \vdash : D \otimes D \to D$, called left and right products.

A dialgebra is called *unital* if it is given a specified bar-unit: an element $1 \in D$ (not necessarily unique) which is a unit for the left and right products only on the bar-side, that is $1 \vdash a = a = a \dashv 1$, for any $a \in D$. A morphism of dialgebras is a K-linear map $f : D \to D'$ which preserves the products, i.e. $f(a \star b) = f(a) \star f(b)$, where $\star$ denotes either the product $\dashv$ or $\vdash$.

Definition 2.2. A dialgebra D over K is called associative if the two operators $\dashv$ and $\vdash$ satisfy the following five axioms*:*

$$\begin{cases} a \dashv (b \dashv c) = (a \dashv b) \dashv c = a \dashv (b \vdash c), \\ (a \vdash b) \dashv c = a \vdash (b \dashv c), \\ (a \vdash b) \vdash c = a \vdash (b \vdash c) = (a \dashv b) \vdash c. \end{cases} \tag{Ass}$$

Obviously an associative dialgebra is an associative algebra if $a \dashv b = a \vdash b = ab$.

Denote by **Dias**, **Ass** the categories of associative dialgebras and associative algebras over K respectively. Then the category **Ass** is a full subcategory of **Dias**.

Definition 2.3. A dialgebra D over K is called alternative if the two operators $\dashv$ and $\vdash$ satisfy the following five axioms:

$$\begin{cases} J_\dashv(a,b,c) = -J_\vdash(c,b,a), \qquad J_\dashv(a,b,c) = J_\vdash(b,c,a), \\ J_\times(a,b,c) = -J_\vdash(a,c,b), \\ (a\vdash b)\vdash c = (a\dashv b)\vdash c, \qquad a\dashv(b\vdash c) = a\dashv(b\dashv c), \end{cases} \tag{Alt}$$

where $J_\dashv(a,b,c) = (a\dashv b)\dashv c - a\dashv(b\dashv c)$, $J_\vdash(a,b,c) = (a\vdash b)\vdash c - a\vdash(b\vdash c)$ and $J_\times(a,b,c) = (a\vdash b)\dashv c - a\vdash(b\dashv c)$.

Obviously, an alternative dialgebra is an alternative algebra if $a\dashv b = a\vdash b = ab$. Moreover, the following formulae are clear for an alternative dialgebras according to the definition:

$$J_\dashv(a,b,c) = -J_\dashv(a,c,b), \tag{2.1}$$

$$J_\vdash(a,b,c) = -J_\vdash(b,a,c), \tag{2.2}$$

$$J_\times(a,b,c) = -J_\times(c,b,a). \tag{2.3}$$

So we also have

$$J_\dashv(a,b,b) = 0, \qquad J_\vdash(a,a,b) = 0, \qquad J_\times(a,b,a) = 0. \tag{2.4}$$

Example.

1. Obviously an associative (alternative) dialgebra is an associative (alternative) algebra if $a\dashv b = a\vdash b = ab$.
2. *Differential associative* (*alternative*) *dialgebra*. Let (A,d) be a differential associative (alternative) algebra. So by hypothesis, $d(ab) = (da)b + a\,db$ and $d^2 = 0$. Define left and right products on A by the formulas

$$x\dashv y = x\,dy, \qquad x\vdash y = (dx)y.$$

   Then A equipped with these two products is an associative (alternative) dialgebra.
3. Let D be an associative (alternative) algebra. On the module of n-space $D = A^{\otimes n}$, one puts

$$(x\dashv y)_i = x_i\left(\sum_{j=1}^n y_j\right) \quad \text{and} \quad (x\vdash y)_i = \left(\sum_{j=1}^n x_j\right)y_i, \quad i = 1,\dots,n.$$

   Then $(D,\dashv,\vdash)$ is an associative (alternative) dialgebra. For $n = 1$, this is Example 1.

A left module over an associative dialgebra D is a K-module M equipped with two linear maps

$$\dashv : D \otimes M \to M, \qquad \vdash : D \otimes M \to M$$

satisfying the axiom (Ass) whenever they make sense. There is, of course, a similar definition for right modules.

A bimodule over an associative dialgebra D, also called a representation, is a K-module M equipped with two linear maps

$$\vdash, \dashv : D \otimes M \to M, \qquad \vdash, \dashv : M \otimes D \to M$$

satisfying the axiom (Ass) whenever they make sense.

Obviously, a bimodule over D is a left module and also a right module, and D is a bimodule over itself.

Let D be an associative dialgebra over K, and M a representation of D. A *derivation* on D with values in M is a K-linear map $\mu : D \to M$ such that

$$\mu(a \star b) = \mu(a) \star b + a \star \mu(b), \quad a, b \in D,$$

where $\star = \dashv, \vdash$. Denote by $\mathrm{Der}(D, M)$ the K-module of all derivations on D with values in M. The *inner derivations* of D in M are the adjoint maps $\mathrm{ad}_m : D \to M$ for $m \in M$ by $\mathrm{ad}_m(x) = x \dashv m - m \vdash x$, for all $x \in D$, and their set is $\mathrm{Inn}(D, M) = \{\mathrm{ad}_m \in \mathrm{Der}(D, M) \mid m \in M\}$.

Given a vector space V over K, the tensor module over V is

$$T(V) = K \oplus V \oplus V^{\otimes 2} \oplus \cdots.$$

The free associative dialgebra is given as follows.

Theorem 2.4 [23]. *The free associative dialgebra on a vector space V is the K-module*

$$\mathrm{Dias}(V) = T(V) \otimes V \otimes T(V)$$

equipped with the two products induced by

$$\begin{aligned}
&(v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m) \dashv (w_{-p} \cdots w_{-1} \otimes w_0 \otimes w_1 \cdots w_q) \\
&\quad = v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m w_{-p} \cdots w_{-1} w_0 w_1 \cdots w_q, \\
&(v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m) \vdash (w_{-p} \cdots w_{-1} \otimes w_0 \otimes w_1 \cdots w_q) \\
&\quad = v_{-n} \cdots v_{-1} v_0 v_1 \cdots v_m w_{-p} \cdots w_{-1} \otimes w_0 \otimes w_1 \cdots w_q,
\end{aligned}$$

where $v_i, w_j \in V$.

For any associative dialgebra D, let D_{Ass} be the quotient of D by the ideal generated by the elements $x \dashv y - x \vdash y$ for all $x, y \in D$. It is clear that D_{Ass} is an associative algebra. The canonical projection $D \to D_{\mathrm{Ass}}$ is universal among the maps from D to associative algebras. In other words the associativization functor $(-)_{\mathrm{Ass}} : \mathbf{Dias} \to \mathbf{Ass}$ is left adjoint to $\mathrm{inc} : \mathbf{Ass} \to \mathbf{Dias}$.

For later use, we recall from [8] the definitions of the modified Hochschild homology of associative dialgebras.

The Hochschild boundary is the K-linear map $d_n : D^{\otimes(n+1)} \to D^{\otimes n}$ given by the formula:

$$\begin{aligned} d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = {} & \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes (a_i \dashv a_{i+1}) \otimes \cdots \otimes a_n \\ & + (-1)^n (a_n \vdash a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned} \tag{2.5}$$

It is easy to check that $d_n \circ d_{n+1} = 0$ (see [8]). Thus we consider the nth homology group $HH_n(D) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$.

Let I be an ideal of D generated by $a \otimes (b \dashv c) - a \otimes (b \vdash c)$, $\forall a, b, c \in D$. Clearly $I \subset \operatorname{Ker} d_1$. Define $C_2(D) = D \otimes D/(\operatorname{Im} d_2 + I)$ and $HHS_1(D) = \operatorname{Ker} d_1/(\operatorname{Im} d_2 + I)$, then we have the following exact sequence:

$$0 \to HHS_1(D) \to C_2(D) \xrightarrow{d_1} D. \tag{2.6}$$

Motivated by Kähler differential modules over commutative associative algebras, we define the Kähler differential modules over commutative associative dialgebras as follows.

For a commutative associative dialgebra D over K (i.e. $a \dashv b = b \vdash a$), the module of differential (Ω_D^1, d) of D is defined in the following way. Let $\{a_i\}$ be any basis for D over K and let F be the free left D-module on a basis $\{\tilde{d}a_i\}$, where $\{\tilde{d}a_i\}$ is some set equipotent with $\{a_i\}$. We treat F as a 2-sided D-module by setting $b \dashv (\tilde{d}a) = (\tilde{d}a) \vdash b$ and $b \vdash (\tilde{d}a) = (\tilde{d}a) \dashv b$ for all $a, b \in D$. Let $\tilde{d} : D \to F$ be the K-linear map $\sum c_i \dashv a_i \mapsto \sum c_i \dashv \tilde{d}a_i$ and let N be the D-submodule generated by the relations $\tilde{d}(a \star b) - (\tilde{d}a) \star b + a \star (\tilde{d}a)$, $a, b \in D$, $\star = \dashv, \vdash$. Then $\Omega_D^1 := F/N$ and the canonical quotient map $a \mapsto \tilde{d}a + K$ is the differential map $d : D \to \Omega_D^1$.

Up to evident isomorphism, (Ω_D^1, d) is characterized by the property that for every D-module M and every derivation $f : D \to M$ there is a unique D-module map $g : \Omega_D^1 \to M$ such that $f = g \circ d$. In this way $\operatorname{Der}_K(D, M) \cong \operatorname{Hom}_D(\Omega_D^1, M)$.

As in the commutative algebra case, we have:

Proposition 2.5. *For a commutative associative dialgebra D, $HHS_1(D) \cong \Omega_D^1$.*

Proof. Define a homomorphism $\zeta : C_2(D) \to \Omega_D^1$ by $\zeta(a \otimes b) = a \dashv db$. It is easy to check that ζ induces an isomorphism $HHS_1(D) \cong \Omega_D^1$. $\square$

2.2. Leibniz algebras

A *Leibniz algebra* [20] L is a vector space over a field K equipped with a K-bilinear map

$$[-,-]: L \times L \to L$$

satisfying the Leibniz identity

$$\bigl[x,[y,z]\bigr] = \bigl[[x,y],z\bigr] - \bigl[[x,z],y\bigr], \quad \forall x,y,z \in L. \tag{2.7}$$

Obviously, a Lie algebra is a Leibniz algebra. A Leibniz algebra is a Lie algebra if and only if $[x,x]=0$ for all $x \in L$.

Suppose that L is a Leibniz algebra over K. For any $z \in L$, we define $\operatorname{ad} z \in \operatorname{End}_k L$ by

$$\operatorname{ad} z(x) = -[x,z], \quad \forall x \in L.$$

It follows (2.7) that

$$\operatorname{ad} z([x,y]) = [\operatorname{ad} z(x), y] + [x, \operatorname{ad} z(y)]$$

for all $x, y \in L$. This says that $\operatorname{ad} z$ is a derivation of L. We also call it the inner derivation of L.

Similarly we also have the definition of general derivation of a Leibniz algebra and we denote by $\operatorname{Inn}(L)$, $\operatorname{Der}(L)$ the set of all inner derivations, derivations of L, respectively. They are also Leibniz algebras.

Let L be a Leibniz algebra, then M is said to be a representation of L if M is a K-vector space equipped with two actions (left and right) of L:

$$[-,-]: L \times M \to M \quad \text{and} \quad [-,-]: M \times L \to M$$

satisfying the following three axioms:

$$\begin{aligned}
\bigl[m,[x,y]\bigr] &= \bigl[[m,x],y\bigr] - \bigl[[m,y],x\bigr],\\
\bigl[x,[m,y]\bigr] &= \bigl[[x,m],y\bigr] - \bigl[[x,y],m\bigr],\\
\bigl[x,[y,m]\bigr] &= \bigl[[x,y],m\bigr] - \bigl[[x,m],y\bigr],
\end{aligned}$$

for any $m \in M$ and $x, y \in L$.

Let L be a Leibniz algebra over K. Consider the boundary map $\delta_n : L^{\otimes n} \to L^{\otimes(n-1)}$ defined by

$$\begin{aligned}
&\delta_n(x_1 \otimes \cdots \otimes x_n)\\
&\quad = \sum_{1 \leqslant i < j \leqslant n} (-1)^{j+1} x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_n,
\end{aligned}$$

where $\hat{x}_j$ indicates that the term x_j is omitted. Clearly $\delta^2 = 0$ [25] and the complex $(L^{\otimes n}, \delta)$ $(L^0 = K, \delta_1 = 0)$ gives the Leibniz homology $HL_*(L)$ of the Leibniz algebra L.

Let L be a Leibniz algebra over K. L is called perfect if $[L, L] = L$. A central extension of L is a pair $(\widehat{L}, \pi)$ where $\widehat{L}$ is a Leibniz algebra and $\pi : \widehat{L} \to L$ is a surjective homomorphism such that $\operatorname{Ker}\pi$ lies in the center of $\widehat{L}$ and the exact sequence $0 \to \operatorname{Ker}\pi \to \widehat{L} \to L \to 0$ splits as K-module. The pair $(\widehat{L}, \pi)$ is a universal central extension of L if for every central extension $(\widetilde{L}, \tau)$ of L there is a unique homomorphism $\psi : \widehat{L} \to \widetilde{L}$ for which $\tau \circ \psi = \pi$. So the universal central extension is unique, up to isomorphism. A Leibniz algebra L has a universal central extension if and only if L is perfect. If $(\widehat{L}, \pi)$ is the universal central extension of L, then

$$HL_2(L) \cong \operatorname{Ker}\pi. \tag{2.8}$$

One has the following proposition.

Proposition 2.6 [16]. *Let $(\widetilde{L}, f)$ and $(\widehat{L}, g)$ be central extensions of a Leibniz algebra L. If $\widehat{L}$ is perfect, then there exists only one homomorphism h from $\widehat{L}$ to $\widetilde{L}$ such that $f \circ h = g$.*

We also denote by **Leib** and **Lie** the categories of Leibniz algebras and Lie algebras over K, respectively.

For any associative dialgebra D, define

$$[x, y] = x \dashv y - y \vdash x,$$

then D equipped with this bracket is a Leibniz algebra. We denote it by D_L. The canonical map $D \mapsto D_L$ induces a functor $(-) : \mathbf{Dias} \to \mathbf{Leib}$.

For a Leibniz algebra L, let L_{Lie} be the quotient of L by the ideal generated by the elements $[x, y] + [y, x]$, for all $x, y \in L$. It is clear that L_{Lie} is a Lie algebra. The canonical projection $L \to L_{\mathrm{Lie}}$ is universal among the maps from L to Lie algebras. In other words, the functor $(-)_{\mathrm{Lie}} : \mathbf{Leib} \to \mathbf{Lie}$ is left adjoint to $\mathrm{inc} : \mathbf{Lie} \to \mathbf{Leib}$.

Moreover, we have the following commutative diagram of categories and functors:

$$\begin{tikzcd}
\mathbf{Dias} \arrow[r, "-"] \arrow[d] & \mathbf{Leib} \arrow[d] \\
\mathbf{Ass} \arrow[r, "-"] & \mathbf{Lie}.
\end{tikzcd}$$

As in the Lie algebra case, the universal enveloping associative dialgebra of a Leibniz algebra L is

$$Ud(L) := \big(T(L) \otimes L \otimes T(L)\big) / \{[x, y] - x \dashv y + y \vdash x \mid x, y \in L\}.$$

Proposition 2.7 [23]. *The functor $Ud : \mathbf{Leib} \to \mathbf{Dias}$ is left adjoint to the functor $- : \mathbf{Dias} \to \mathbf{Leib}$.*

3. Steinberg–Leibniz algebras

The matrix Leibniz algebra $\mathfrak{gl}(n, D)$ is generated by all $n \times n$ matrices with coefficients from a unital associative dialgebra D, and $n \geqslant 3$ with the bracket

$$[E_{ij}(a), E_{kl}(b)] = \delta_{jk} E_{il}(a \dashv b) - \delta_{il} E_{kj}(b \vdash a),$$

for all $a, b \in D$, where $E_{ij}(a)$ is the $n \times n$ matrix with coefficient a on (i, j)-th position and 0 in all others.

Clearly, $\mathfrak{gl}(n, D)$ is a Leibniz algebra. If D is an associative algebra, then $\mathfrak{gl}(n, D)$ becomes a Lie algebra.

Now we consider the subalgebra $\mathfrak{sl}(n, D) := [\mathfrak{gl}(n, D), \mathfrak{gl}(n, D)]$, which is called the *special linear Leibniz algebra* with coefficients in D, of $\mathfrak{gl}(n, D)$.

By definition, the special linear Leibniz algebra $\mathfrak{sl}(n, D)$ has generators $E_{ij}(a)$, $1 \leqslant i \neq j \leqslant n$, $a \in D$, which satisfy the following relations:

$$[E_{ij}(a), E_{kl}(b)] = 0, \quad \text{if } i \neq l \text{ and } j \neq k;$$

$$[E_{ij}(a), E_{kl}(b)] = E_{il}(a \dashv b) \quad \text{if } i \neq l \text{ and } j = k;$$

$$[E_{ij}(a), E_{kl}(b)] = -E_{kj}(b \vdash a), \quad \text{if } i = l \text{ and } j \neq k.$$

Let D be a unital dialgebra (not necessary associative), we define the *Steinberg–Leibniz algebra* $\mathfrak{stl}(n, D)$, which is a Leibniz algebra generated by symbols $v_{ij}(a)$, $1 \leqslant i \neq j \leqslant n$, $a \in D$, subject to the relations:

(1) $v_{ij}(k_1 a + k_2 b) = k_1 v_{ij}(a) + k_2 v_{ij}(b), \quad$ for $a, b \in D,\ k_1, k_2 \in K$;

(2) $[v_{ij}(a), v_{kl}(b)] = 0, \quad$ if $i \neq l$ and $j \neq k$;

(3) $[v_{ij}(a), v_{kl}(b)] = v_{il}(a \dashv b), \quad$ if $i \neq l$ and $j = k$;

(4) $[v_{ij}(a), v_{kl}(b)] = -v_{kj}(b \vdash a), \quad$ if $i = l$ and $j \neq k$.

It is clear that the last two relations make sense only if $n \geqslant 3$.

Let $H_{ij}(a, b) := [v_{ij}(a), v_{ji}(b)]$ for $1 \leqslant i \neq j \leqslant n$, $a, b \in D$, and H be the submodule of $\mathfrak{stl}(n, D)$ generated by the elements $H_{ij}(a, b)$, $i \neq j$, $a, b \in D$.

Define $i_n(D) = \{a \in D \mid v_{ij}(a) = 0\}$. It is an ideal of D, which does not depend on the choice of $i \neq j$. By similar considerations as in the Steinberg–Lie algebra case, we get

Proposition 3.1. *For a unital dialgebra D, $i_n(D) = 0$ in $\mathfrak{stl}(n, D)$ if and only if D is associative for $n \geqslant 4$ and D is alternative for $n = 3$.*

Proof. Suppose that $i_n(D) = 0$, we shall prove that D satisfies the axiom (Ass) for $n \geqslant 4$ and D satisfies the axiom (Alt) for $n = 3$.

For $\{i, j\} \neq \{k, t\}$, the following formulae are immediate:

$$
\begin{aligned}
&[H_{ij}(r,s), v_{kt}(u)] \\
&\quad = X_{kt}\big(\delta_{ik} r \dashv (s \dashv u) + \delta_{jt}(u \vdash s) \vdash r - \delta_{jk} s \vdash (r \dashv u) - \delta_{it}(u \vdash r) \dashv s\big). \qquad (3.1)\\
&[v_{kt}(u), H_{ij}(r,s)] \\
&\quad = X_{kt}\big(-\delta_{ik} r \vdash (s \vdash u) - \delta_{jt}(u \dashv s) \dashv r + \delta_{jk} s \vdash (r \vdash u) + \delta_{it}(u \dashv r) \dashv s\big). \qquad (3.2)
\end{aligned}
$$

Consider the case $\{i, j\} = \{k, t\}$ and choose $m \notin \{i, j\}$; then

$$
\begin{aligned}
[H_{ij}(r,s), v_{ij}(u \dashv v)] &= \big[H_{ij}(r,s), [v_{im}(u), v_{mj}(v)]\big] \\
&= \big[[H_{ij}(r,s), v_{im}(u)], v_{mj}(v)\big] - \big[[H_{ij}(r,s), v_{mj}(v)], v_{im}(u)\big] \\
&= v_{ij}\big(\big(r \dashv (s \dashv u)\big) \dashv v + u \vdash \big((v \vdash s) \vdash r\big)\big). \qquad (3.3)
\end{aligned}
$$

Now for $n \geqslant 3$, we choose distinct i, j, l. By

$$\big[v_{il}(r), [v_{ij}(u), v_{li}(s)]\big] = -\big[[v_{il}(r), v_{li}(s)], v_{ij}(u)\big] = -[H_{il}(r,s), v_{ij}(u)]$$

and (3.1), we get

$$r \dashv (s \vdash u) = r \dashv (s \dashv u).$$

Similarly, we obtain $(r \dashv s) \vdash u = (r \vdash s) \vdash u$.

If $n \geqslant 4$, we can choose distinct i, j, k, l, then the axiom (Ass) of D follows from the following formulas:

$$
\begin{aligned}
\big[v_{ij}(r), [v_{jk}(s), v_{kl}(u)]\big] &= \big[[v_{ij}(r), v_{jk}(s)], v_{kl}(u)\big], \\
\big[[v_{ij}(r), v_{ki}(s)], v_{jl}(u)\big] &= \big[[v_{ij}(r), v_{jl}(u)], v_{ki}(s)\big], \\
\big[v_{ij}(r), [v_{ki}(s), v_{lk}(u)]\big] &= \big[[v_{ij}(r), v_{ki}(s)], v_{lk}(u)\big].
\end{aligned}
$$

If $n = 3$, we replace u by 1 and v by $u \dashv v$ in (3.3); then

$$\big(r \dashv (s \dashv u)\big) \dashv v + u \vdash \big((v \vdash s) \vdash r\big) = (r \dashv s) \dashv (u \dashv v) + \big((u \vdash v) \vdash s\big) \vdash r. \qquad (3.4)$$

Setting $v = 1$ in (3.4), we obtain $J_{\dashv}(r, s, u) = -J_{\vdash}(u, s, r)$. Setting $s = 1$ in (3.4), we obtain $J_{\dashv}(r, u, v) = J_{\vdash}(u, v, r)$. By (3.1) we have

$$[H_{ij}(r,s), v_{kj}(u \dashv v)] = v_{kj}\big(\big((u \dashv v) \vdash s\big) \vdash r\big).$$

But

$$
\begin{aligned}
[H_{ij}(r,s), v_{kj}(u \dashv v)] &= \big[H_{ij}(r,s), [v_{ki}(u), v_{ij}(v)]\big] \\
&= \big[[H_{ij}(r,s), v_{ki}(u)], v_{ij}(v)\big] - \big[[H_{ij}(r,s), v_{ij}(v)], v_{ki}(u)\big] \\
&= v_{kj}\big(-\big((u \vdash r) \dashv s\big) \dashv v + u \vdash \big(r \dashv (s \dashv v)\big) + u \vdash \big(v \vdash (s \vdash r)\big)\big).
\end{aligned}
$$

So we have

$$\big((u \dashv v) \vdash s\big) \vdash r + \big((u \vdash r) \dashv s\big) \dashv v = u \vdash \big(r \dashv (s \dashv v)\big) + u \vdash \big(v \vdash (s \vdash r)\big). \quad (3.5)$$

Setting $v = 1$ in (3.5), we get

$$(u \vdash s) \vdash r + (u \vdash r) \dashv s = u \vdash (r \dashv s) + u \vdash (s \vdash r), \quad \text{i.e.,} \quad J_\times(u, r, s) = -J_\vdash(u, s, r).$$

The opposite proof is similar to that given in [7]. □

Remarks. 1. If D is an associative algebra and $n \geqslant 3$ ($\dashv = \vdash$), then $\mathfrak{sl}(n, D)$ is just the special linear Lie algebra, but $\mathfrak{stl}(n, D)$ is the Leibniz algebra defined in [12,25].

2. Let D be a unital dialgebra. If D is alternative, $\mathfrak{sl}(3, D)$ is not a Leibniz algebra in general. In fact $\mathfrak{sl}(3, D)$ is a Leibniz algebra if and only if D is associative.

From now always suppose that D is a unital associative dialgebra in $\mathfrak{stl}(n, D)$ and $\mathfrak{sl}(n, D)$ for $n \geqslant 3$ and then $i_n(D) = 0$.

With the axiom (Ass) in Section 2, we get

$$[v_{ij}(a), v_{kl}(b \dashv c)] = [v_{ij}(a), v_{kl}(b \vdash c)], \quad \forall a, b, c \in D. \quad (3.6)$$

Now we define a homomorphism: $\psi : \mathfrak{stl}(n, D) \to \mathfrak{sl}(n, D)$ of Leibniz algebras by the rule $\psi(u_{ij}(a)) = E_{ij}(a)$. It is clear that φ is surjective.

Lemma 3.2. *The Steinberg–Leibniz algebra* $\mathfrak{stl}(n, D)$ *with* $n \geqslant 3$ *is perfect.*

Denote by P (respectively Q) the K-submodule of $\mathfrak{stl}(n, A)$ generated by $v_{ij}(a)$ with $i < j$ (respectively $i > j$). It is clear that: the restrictions of ψ to P and Q are injective; the images of $v_{ij}(a)$ from P (respectively Q) under ψ are upper triangular (respectively lower triangular) matrices in $\mathfrak{sl}(n, D)$ with 0 on the main diagonal.

Lemma 3.3. *Every element* $X \in \mathfrak{stl}(n, D)$ *with* $n \geqslant 3$ *can be uniquely written in the form*

$$X = p + h + q, \quad \textit{where } p \in P,\ h \in H,\ q \in Q. \quad (3.7)$$

Proof. We show that $\mathfrak{stl}(n, D)$ is a direct sum of the K-submodules P, H, and Q.

At first, we note that $P + H + Q$ is an ideal of $\mathfrak{stl}(n, D)$. It follows from the fact that for any $v_{ij}(a)$ and $X \in P + H + Q$, we have $[v_{ij}(a), X] \in P + H + Q$. Indeed, we need to check this only for $X = v_{kl}(b)$, $k \neq l$ and for $X = H_{kl}(b, c)$, $k \neq l$. But this holds by the definition of $\mathfrak{stl}(n, D)$ and direct calculations.

To check the uniqueness part of decomposition (3.7), we suppose that

$$p + h + q = 0, \quad p \in P,\ h \in H,\ q \in Q.$$

Then $\psi(p+h+q)=0$, i.e., $\psi(p)+\psi(h)+\psi(q)=0$. But $\psi(p)$ is the upper triangular of $\psi(p+h+q)=0$. Therefore $\psi(p)=0$. But by injectivity of ψ, we get $p=0$. Analogously, we have $q=0$, and thus $h=0$ since $\psi(h)$ is diagonal. $\square$

Lemma 3.4. *For $n \geqslant 3$, $\operatorname{Ker}\psi \subseteq H$.*

Proof. Let $X=p+h+q \in \operatorname{Ker}\psi$, where $p \in P$, $h \in H$, $q \in Q$. Then $0=\psi(X)=\psi(p)+\psi(h)+\psi(q)$. By Lemma 3.3, we have $\psi(p)=\psi(q)=0$ and therefore, $p=q=0$. So $X=h \in H$. $\square$

4. The universal central extension of $\mathfrak{sl}(n, D)$

In the following, D always means a unital associative dialgebra over K.

Theorem 4.1. *If $n \geqslant 3$, then $(\mathfrak{stl}(n, D), \psi)$ is a central extension of the Leibniz algebra $\mathfrak{sl}(n, D)$.*

Proof. We only need to show that $[\operatorname{Ker}\psi, \mathfrak{stl}(n, D)]=0$ for $n \geqslant 3$.

By Lemma 3.4, any element $t \in \operatorname{Ker}\psi$ is expressible by $H_{kl}(b,c)$'s. By (2.7), $[v_{ij}(a), H_{kl}(b,c)] \in P+Q$. Therefore, we have $[v_{ij}(a), t]=p+q$, where $p \in P$, $q \in Q$. Thus $\psi(p+q)=\psi([v_{ij}(a), t])=[\psi(v_{ij}(a)), \psi(t)]=0$ since $\psi(t)=0$. By injectivity of the restriction of ψ to $P+Q$, we get $p+q=0$. So $[\operatorname{Ker}\psi, \mathfrak{stl}(n, D)]=0$. $\square$

Now we show that the central extension $(\mathfrak{stl}(n, D), \psi)$ is in fact universal.

Theorem 4.2. *Let (W, ϕ) be a central extension of the Leibniz algebra $\mathfrak{sl}(n, D)$, and $n \geqslant 3$ with $\operatorname{char} K \neq 3$ if $n=3$ and $\operatorname{char} K \neq 2$ if $n=4$. Then there exists a unique homomorphism $\rho : \mathfrak{stl}(n, D) \to W$ such that $\phi \circ \rho = \psi$.*

Proof. For $n=3$ and $\operatorname{char} K \neq 3$, we can use the same method in the proof of Theorem 5.18 in [1] to prove that $\mathfrak{stl}(n, D)$ is centrally closed. The differences are that we set $\mathcal{M}=v_{13}(D) \oplus v_{31}(D) \oplus v_{23}(D) \oplus v_{32}(D)$ and the derivation $\mathcal{D}=\operatorname{ad}(v_{12}+v_{21}(1))$ here and $\mathcal{D}|_{\mathcal{M}}$ is diagonalizable with eigenvalues ± 1.

Now we prove Theorem 4.2 for $n \geqslant 4$.

Since $\phi : W \to \mathfrak{sl}(n, D)$ is surjective, for any generator $E_{ij}(a) \in \mathfrak{sl}(n, D)$, we choose $e_{ij}(a) \in \phi^{-1}(E_{ij}(a))$. Then the commutator $[e_{ij}(a), e_{kl}(b)]$ does not depend on the choice of representatives of $\phi^{-1}(E_{ij}(a))$ and $\phi^{-1}(E_{kl}(b))$. Moreover, for any $j \neq k$ and $i \neq l$ and $a, b \in D$, $\phi([e_{ij}(a), e_{kl}(b)])=[\phi(e_{ij}(a)), \phi(e_{kl}(b))]=0$. So $[e_{ij}(a), e_{kl}(b)] \in \operatorname{Ker}\phi$.

For distinct i, j, k, let

$$[e_{ik}(a), e_{kj}(b)]=e_{ij}(a \dashv b)+C_{ij}^{k}(a,b), \tag{4.1}$$

where $C_{ij}^{k}(a,b) \in \operatorname{Ker}\phi$. Take $l \notin \{i, j, k\}$, then

$$
\begin{aligned}
[e_{ik}(a), e_{kj}(b \dashv c)] &= \big[e_{ik}(a), [e_{kl}(b), e_{lj}(c)] + C_{ij}^{k}(a, b \dashv c)\big] = \big[e_{ik}(a), [e_{kl}(b), e_{lj}(c)]\big] \\
&= \big[[e_{ik}(a), e_{kl}(b)], e_{lj}(c)\big] - \big[[e_{ik}(a), e_{lj}(c)], e_{kl}(b)\big] \\
&= [e_{il}(a \dashv b), e_{lj}(c)], \\
[e_{ik}(a), e_{kj}(b \vdash c)] &= \big[e_{ik}(a), -[e_{lj}(c), e_{kl}(b)]\big] \\
&= -\big[[e_{ik}(a), e_{lj}(c)], e_{kl}(b)\big] + \big[[e_{ik}(a), e_{kl}(b)], e_{lj}(c)\big] \\
&= [e_{il}(a \dashv b), e_{lj}(c)], \\
[e_{kj}(b \dashv c), e_{ik}(a)] &= \big[[e_{kl}(b), e_{lj}(c)], e_{ik}(a)\big] \\
&= \big[e_{kl}(b), [e_{lj}(c), e_{ik}(a)]\big] + \big[[e_{kl}(b), e_{ik}(a)], e_{lj}(c)\big] \\
&= -[e_{il}(a \vdash b), e_{lj}(c)].
\end{aligned}
$$

So

$$[e_{ik}(a), e_{kj}(b \dashv c)] = [e_{il}(a \dashv b), e_{lj}(c)] = [e_{ik}(a), e_{kj}(b \dashv c)] \tag{4.2}$$

and

$$[e_{kj}(b \dashv c), e_{ik}(a)] = -[e_{il}(a \vdash b), e_{lj}(c)]. \tag{4.3}$$

Taking $b = 1$ in (4.2), we have

$$[e_{ik}(a), e_{kj}(c)] = [e_{il}(a), e_{lj}(c)]. \tag{4.4}$$

It follows that $C_{ij}^{k}(a, c) = C_{ij}^{l}(a, c)$ which show that C_{ij}^{k} is independent of the choice of k. Setting $C_{ij}^{k}(a, b) = C_{ij}(a, b)$, we have

$$[e_{ik}(a), e_{kj}(b)] = e_{ij}(a \dashv b) + C_{ij}(a, b), \tag{4.5}$$

where $C_{ij}(a, b) \in \operatorname{Ker}\phi$. Taking $b = 1$ we have

$$[e_{ik}(a), e_{kj}(1)] = e_{ij}(a) + C_{ij}(a, 1). \tag{4.6}$$

Now we replace $e_{ij}(a)$ by $e_{ij}(a) + C_{ij}(a, 1)$; then we have

$$[e_{ik}(a), e_{kj}(1)] = e_{ij}(a). \tag{4.7}$$

Taking $c = 1$ in (4.2) and using (4.7), we get

$$[e_{ik}(a), e_{kj}(b)] = [e_{ik}(a \dashv b), e_{kj}(1)] = e_{ij}(a \dashv b). \tag{4.8}$$

But taking $c = 1$ in (4.3),

$$[e_{kj}(b), e_{ik}(a)] = -[e_{il}(a \vdash b), e_{lj}(1)] = -e_{ij}(a \dashv b). \tag{4.9}$$

Now we shall show that

$$[e_{ij}(a), e_{kl}(b)] = 0, \quad j \neq k \text{ and } i \neq l. \tag{4.10}$$

If $n \geqslant 5$, $n = 4$ but $i = k$ or $j = l$, we may choose $s \notin \{i, j, k, l\}$. Then $e_{kl}(b) = [e_{ks}(b), e_{sl}(1)]$. Hence it follows that

$$\begin{aligned}
[e_{ij}(a), e_{kl}(b)] &= \big[e_{ij}(a), [e_{ks}(b), e_{sl}(1)]\big] \\
&= \big[[e_{ij}(a), e_{ks}(b)], e_{sl}(1)\big] - \big[e_{ks}(b), [e_{ij}(a), e_{sl}(1)]\big] \\
&= 0.
\end{aligned}$$

If $n = 4$ and i, j, l, k is distinct,

$$\begin{aligned}
[e_{ij}(a \dashv b), e_{kl}(c)] &= \big[[e_{ik}(a), e_{kj}(b)], e_{kl}(c)\big] = \big[[e_{ik}(a), e_{kl}(c)], e_{kj}(b)\big] \\
&= [e_{il}(a \dashv c), e_{kj}(b)], \\
[e_{ij}(a \dashv b), e_{kl}(c)] &= \big[[e_{il}(a), e_{lj}(b)], e_{kl}(c)\big] = \big[e_{il}(a), [e_{lj}(b), e_{kl}(c)]\big] \\
&= -[e_{il}(a), e_{kj}(c \vdash b)].
\end{aligned}$$

So we have

$$[e_{ij}(a \dashv b), e_{kl}(c)] = [e_{il}(a \dashv c), e_{kj}(b)] = -[e_{il}(a), e_{kj}(c \vdash b)]. \tag{4.11}$$

Taking $b = c = 1$ in (4.11), we have

$$[e_{ij}(a), e_{kl}(1)] = 0. \tag{4.12}$$

Taking $b = 1$ in (4.11) and with (4.12), we obtain (4.10).

Now we see that $e_{ij}(a)$, $1 \leqslant i \neq j \leqslant n$, $a \in \mathcal{A}$, satisfy the relations (1)–(4) in the definition of $\mathfrak{stl}(n, D)$. Then there is a homomorphism $\rho : \mathfrak{stl}(n, D) \to W$ defined by

$$\rho\big(v_{ij}(a)\big) = e_{ij}(a).$$

From the above, we see that this mapping is well defined and $\phi \circ \rho = \psi$.

The uniqueness of the mapping ρ follows from Proposition 2.6 and Lemma 3.2. $\square$

Remark. In (4.12) we use the restriction $\operatorname{char} K \neq 2$. From the proof of Theorem 5.18 in [1] we must restrict $\operatorname{char} K \neq 3$ if $n = 3$.

Corollary 4.3. *For a unital associative dialgebra D, the Steinberg–Leibniz algebra $\mathfrak{stl}(n, D)$ is the universal central extension of $\mathfrak{sl}(n, D)$ in the category of Leibniz algebras when $n \geqslant 3$ with* $\operatorname{char} K \neq 3$ *if* $n = 3$ *and* $\operatorname{char} K \neq 2$ *if* $n = 4$.

5. Kernel of the universal central extension $(\mathfrak{stl}(n, D), \psi)$

Now we can prove the following theorem.

Theorem 5.1. *The kernel $T(n)$ of the universal central extension $(\mathfrak{stl}(n, D), \psi)$ of $\mathfrak{sl}(n, D)$ is isomorphic to $HHS_1(D)$ for $n \geqslant 3$ (with* char $K \neq 3$ *if $n = 3$ and* char $K \neq 2$ *if $n = 4$).*

This theorem follows directly from the following lemmas.

Lemma 5.2. *For H_{ij}, we have*:

(a) *the mapping $(a, b) \to H_{ij}(a, b)$ is K-bilinear*;
(b) $H_{ij}(a, b \vdash c) = H_{ik}(a \dashv b, c) + H_{kj}(c \vdash a, b) = H_{ij}(a, b \dashv c)$;
(c) $H_{ij}(a, 1) = -H_{ji}(a, 1)$, *for any distinct i, j.*

Proof. (a) It is clear.

$$\text{(b)}\quad \begin{aligned} H_{ij}(a, b \dashv c) &= [v_{ij}(a), v_{ji}(b \dashv c)] = \big[v_{ij}(a), [v_{jk}(b), v_{ki}(c)]\big] \\ &= \big[[v_{ij}(a), v_{jk}(b)], v_{ki}(c)\big] - \big[[v_{ij}(a), v_{ki}(c)], v_{jk}(b)\big] \\ &= [v_{ik}(a \dashv b), v_{ki}(c)] + [v_{kj}(c \vdash a), v_{jk}(b)] \\ &= H_{ik}(a \dashv b, c) + H_{kj}(c \vdash a, b). \end{aligned}$$

The second equality of (b) is due to (3.6).

(c) We put $b = c = 1$ in (b) and get

$$H_{ij}(a, 1) = H_{ik}(a, 1) + H_{kj}(a, 1).$$

Changing k, j in the above formula, we have

$$H_{ik}(a, 1) = H_{ij}(a, 1) + H_{jk}(a, 1).$$

So $H_{kj}(a, 1) = -H_{jk}(a, 1)$. $\square$

Lemma 5.3. *For any j, k,*

$$H_{1j}(a, b) - H_{1j}(b \vdash a, 1) = H_{1k}(a, b) - H_{1k}(b \vdash a, 1).$$

Proof.

$$\begin{aligned} &H_{1j}(a, b) - H_{1j}(b \vdash a, 1) \\ &\quad = H_{1j}(a, 1 \vdash b) - H_{1j}(b \vdash a, 1 \vdash 1) \\ &\quad = H_{1k}(a \dashv 1, b) + H_{kj}(b \vdash a, 1) - \big(H_{1k}(b \vdash a, 1) + H_{kj}(b \vdash a, 1)\big) \\ &\quad = H_{1k}(a, b) - H_{1k}(b \vdash a, 1). \qquad \square \end{aligned}$$

With Lemma 5.3, it is possible to define the elements

$$h(a,b)=H_{1j}(a,b)-H_{1j}(b\vdash a,1),$$

which does not depend on j ($\neq 1$).

Lemma 5.4. (a) *The mapping* $(a,b)\mapsto h(a,b)$ *is* K*-bilinear.*
(b) $h(a,b\dashv c)=h(a\dashv b,c)+h(c\vdash a,b)=h(a,b\vdash c)$.
(c) $h(a,1)=0$.

Proof. The statement (a) directly follows from the definition of $h(a,b)$ and Lemma 5.2.
To prove (b), we note that

$$h(a,b\dashv c)=H_{1j}(a,b\dashv c)-H_{1j}\big((b\dashv c)\vdash a,1\big). \tag{5.1}$$

By (5.1), we get

$$\begin{aligned}
h(a,b\dashv c)&=H_{1j}(a,b\dashv c)-H_{1j}\big((b\dashv c)\vdash a,1\big)\\
&=H_{1k}(a\dashv b,c)+H_{kj}(c\vdash a,b)-H_{1j}\big((b\dashv c)\vdash a,1\big)\\
&=\big(H_{1k}(a\dashv b,c)-H_{1k}\big(c\vdash(a\dashv b),1\big)\big)\\
&\quad+\big(H_{1k}\big(c\vdash(a\dashv b),1\big)+H_{kj}(c\vdash a,b)\big)-H_{1j}\big((b\dashv c)\vdash a,1\big)\\
&=h(a\dashv b,c)+\big(H_{1k}\big((c\vdash a)\dashv b\big),1\big)+H_{kj}\big(1\vdash(c\vdash a),b\big)\\
&\quad-H_{1j}\big((b\dashv c)\vdash a,1\big)\\
&=h(a\dashv b,c)+H_{1j}(c\vdash a,b\vdash 1)-H_{1j}\big(b\dashv(c\vdash a),1\big)\\
&=h(a\dashv b,c)+H_{1j}(c\vdash a,b\dashv 1)-H_{1j}\big(b\dashv(c\vdash a),1\big)\\
&=h(a\dashv b,c)+H_{1j}(c\vdash a,b)-H_{1j}\big(b\dashv(c\vdash a),1\big)\\
&=h(a\dashv b,c)+h(c\vdash a,b).
\end{aligned}$$

The second equality is due to (b) in Lemma 5.2 and the axiom (Ass).
(c) It is clear. □

To prove that $T(n)$ and $HHS_1(D)$ are isomorphic, we consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & HHS_1(D) \arrow[r] & C_2(D) \arrow[r, "d_1"] \arrow[d, "\eta"] & D \arrow[d, "\mu"] \\
0 \arrow[r] & T(n) \arrow[r] & \mathfrak{stl}(n,D) \arrow[r, "\psi"] & \mathfrak{gl}(n,D)
\end{tikzcd} \tag{5.2}$$

where the mappings μ and η are defined by $\mu(a)=E_{11}(a)$ and $\eta(a\otimes b)=h(a,b)$. It follows from Lemma 5.4 that the mapping η is well defined.

Lemma 5.5. *The diagram* (5.2) *is commutative and the restriction to $HHS_1(D)$ of η is a surjective homomorphism onto the kernel $T(n)$.*

Proof. By direct calculation, we can obtain the commutativity of diagram (5.2):

$$\begin{aligned}
\psi \circ \eta(a \otimes b) &= \psi\big(h(a,b)\big) = \psi\big(H_{1j}(a,b)\big) - \psi\big(H_{1j}(b \vdash a, 1)\big) \\
&= \psi\big([v_{1j}(a), v_{j1}(b)]\big) - \psi\big(v_{1j}(b \vdash a), [v_{j1}(1)]\big) \\
&= \big[\psi\big(v_{1j}(a)\big), \psi\big(v_{j1}(b)\big)\big] - \big[\psi\big(v_{1j}(b \vdash a)\big), \psi\big([v_{j1}(1)]\big)\big] \\
&= [E_{1j}(a), E_{j1}(b)] - [E_{1j}(b \vdash a), E_{j1}(1)] \\
&= E_{11}(a \dashv b) - E_{jj}(b \vdash a) - E_{11}(b \vdash a) + E_{jj}(b \vdash a) \\
&= E_{11}(a \dashv b) - E_{11}(b \vdash a) = E_{11}(a \dashv b - b \vdash a) = E_{11}\big(d_1(a \otimes b)\big) \\
&= \mu \circ d_1(a \otimes b).
\end{aligned}$$

Now we prove that η is surjective.

If $a \otimes b \in HHS_1(D)$, then $d_1(a \otimes b) = 0$, i.e., $a \dashv b = b \vdash a$. So $\psi(h(a,b)) = E_{11}(a \dashv b - b \vdash a) = 0$. Therefore $h(a,b) \in T(n)$, i.e., for the restriction of η to $HHS_1(D)$ we have a mapping into the kernel $T(n)$.

For any element $t \in T(n)$, $t \in H$ by Lemma 3.4, where H is the K-module generated by $H_{ij}(a,b)$, $i \neq j$, $a, b \in D$. Taking into account the properties of (b), (c) of Lemma 5.2, any generator $H_{ij}(a)$ can be written in the following way:

- If $i = 1$, then

$$H_{1j}(a,b) = H_{1j}(a,b) - H_{1j}(b \vdash a, 1) + H_{1j}(b \vdash a, 1) = h(a,b) + H_{1j}(b \vdash a, 1).$$

- If $i \neq 1$, we set $k = 1$ in (b) of Lemma 5.2 and get:

$$\begin{aligned}
H_{ij}(a,b) &= H_{ij}(a, 1 \vdash b) = H_{i1}(a,b) + H_{1j}(b \vdash a, 1) \\
&= -H_{1i}(a,b) + H_{1j}(b \vdash a, 1) = -h(a,b) - H_{1i}(b \vdash a, 1) + H_{1j}(b \vdash a, 1).
\end{aligned}$$

Thus any element $t \in T(n)$ has a presentation in the form

$$t = \sum_i h(a_i, b_i) + \sum_{j \geqslant 2} H_{1j}(c_j, 1).$$

Since $\psi(t) = 0$, we have

$$\begin{aligned}
\psi(t) &= \psi\left(\sum_i h(a_i, b_i) + \sum_{j \geqslant 2} H_{1j}(c_j, 1)\right) \\
&= \sum_i \psi(h(a_i, b_i)) + \sum_{j \geqslant 2} \psi\big(H_{1j}(c_j, 1)\big)
\end{aligned}$$

$$= E_{11}\Biggl(\sum_i (a_i \dashv b_i - b_i \vdash a_i)\Biggr) + \sum_{j\geqslant 2}\bigl(E_{11}(c_j) - E_{jj}(c_j)\bigr)$$
$$= 0.$$

So we have $c_j = 0$, hence for all $j \geqslant 2$ we get $H_{1j}(c_j, 1) = 0$ and

$$t = \sum_i h(a_i, b_i) = \eta\Biggl(\sum_i a_i \otimes b_i\Biggr).$$

So the restriction to $HHS_1(D)$ of η is a surjective homomorphism onto the kernel $T(n)$. $\square$

Let $M_n(D)$ be the K-algebra of $(n \times n)$-matrices with coefficients from D, and $n \geqslant 1$. Now we define a homomorphism $\mathrm{tr}_2 : M_n(D) \otimes M_n(D) \to D \otimes D$ by

$$\mathrm{tr}_2(P \otimes Q) = \sum_{1\leqslant i,j\leqslant l} p_{ij} \otimes q_{ji}, \quad \text{for } P, Q \in M_n(D). \tag{5.3}$$

To prove injectivity of η, we consider the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & T(m) & \longrightarrow & \mathfrak{stl}(n, D) & \xrightarrow{\psi} & \mathfrak{gl}(n, D) \\
 & & & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\mathrm{tr}} \\
0 & \longrightarrow & HHS_1(D) & \longrightarrow & C_2(D) & \xrightarrow{d_1} & D.
\end{array} \tag{5.4}$$

Lemma 5.6. *There exists a homomorphism of K-modules $\theta : \mathfrak{stl}(n, D) \to C_2(D)$ such that the diagram* (5.4) *is commutative and $\theta \circ \eta$ is the identical mapping on $C_2(D)$.*

Proof. Define the mapping θ in a similar way as in [13]:

(1) θ is a homomorphism of K-modules $\theta : \mathfrak{stl}(n, D) \to C_2(D)$;
(2) $\theta([x, y]) = \sum_{i,j} \psi(x)_{ij} \otimes \psi(y)_{ji} = \mathrm{tr}_2(\psi(x) \otimes \psi(y))$;
(3) $\theta(v_{ij}(a)) = 0$. Then

$$\theta(H_{ij}(a, b)) = \theta\bigl([v_{ij}(a), v_{ji}(b)]\bigr) = \mathrm{tr}_2\bigl(\psi\bigl(v_{ij}(a)\bigr) \otimes \psi\bigl(v_{ji}(b)\bigr)\bigr)$$
$$= \mathrm{tr}_2\bigl(E_{ij}(a) \otimes E_{ji}(b)\bigr) = a \otimes b.$$

So $\theta(H_{ij}(a, b)) \in C_2(D)$ by (b) in Lemma 5.2.

Moreover,

$$\theta\bigl([v_{ij}(a), v_{kl}(b)]\bigr) = \mathrm{tr}_2\bigl(\psi\bigl(v_{ij}(a)\bigr) \otimes \psi\bigl(v_{kl}(b)\bigr)\bigr) = \mathrm{tr}_2\bigl(E_{ij}(a) \otimes E_{kl}(b)\bigr)$$

$$= \begin{cases} 0, & \text{if } i \neq l,\ j \neq k; \\ \theta\big(v_{il}(a \dashv b)\big), & \text{if } i \neq l,\ j = k; \\ \theta\big(-v_{kj}(b \vdash a)\big), & \text{if } i = l,\ j \neq k. \end{cases}$$

To prove the commutativity of the diagram (5.4), it is sufficient to check it on elements v_{ij} and $H_{ij}(a,b)$ from $\mathfrak{stl}(n,D)$. Indeed:

$$\begin{aligned} d_1 \circ \theta\big(v_{ij}(a)\big) &= 0; \\ \operatorname{tr} \circ \psi\big(v_{ij}(a)\big) &= \operatorname{tr}\big(E_{ij}(a)\big) = 0, \quad \text{since } i \neq j; \\ d_1 \circ \theta\big(H_{ij}(a,b)\big) &= d_1(a \otimes b) = a \dashv b - b \vdash a = \operatorname{tr} \circ \psi\big(H_{ij}(a,b)\big) \\ &= \operatorname{tr}\big(E_{ii}(a \dashv b) - E_{jj}(b \vdash a)\big) = a \dashv b - b \vdash a. \end{aligned}$$

Now we show that $\theta \circ \eta$ is identical on $C_2(D)$:

$$\begin{aligned} \theta \circ \eta(a \otimes b) &= \theta\big(h(a,b)\big) = \theta\big(H_{1j}(a,b) - H_{1j}(b \vdash a, 1)\big) \\ &= \theta\big(H_{1j}(a,b)\big) - \theta\big(H_{1j}(b \vdash a, 1)\big) = a \otimes b - (b \vdash a) \otimes 1 \\ &\equiv a \otimes b \pmod{\operatorname{Im} d_2}, \end{aligned}$$

since

$$d_2(a \otimes 1 \otimes b) = (a \dashv 1) \otimes b - a \otimes (1 \dashv b) + (b \vdash a) \otimes 1 = (b \vdash a) \otimes 1$$

as $a \otimes (1 \dashv b) = a \otimes (1 \vdash b) = a \otimes b$ in $C_2(D)$. $\square$

By Theorem 5.1 and (2.8), we have:

Corollary 5.7. *When D is a unital associative dialgebra, $n \geqslant 3$ with $\operatorname{char} K \neq 3$ if $n = 3$ and $\operatorname{char} K \neq 2$ if $n = 4$,*

$$HL_2\big(\mathfrak{sl}(n,D)\big) = HHS_1(D).$$

If D is a unital commutative associative dialgebra, we have the following corollary by Proposition 2.5.

Corollary 5.8. *When D is a unital commutative associative dialgebra,*

$$0 \to \Omega_D^1 \to \mathfrak{stl}(n,D) \to \mathfrak{sl}(n,D) \to 0$$

is the universal central extension of $\mathfrak{sl}(n,D)$ for $n \geqslant 3$ with $\operatorname{char} K \neq 3$ if $n = 3$ and $\operatorname{char} K \neq 2$ if $n = 4$.

Remarks. 1. If D is a unital associative algebra, we obtain the same results as in [25] and [12].

2. The stable version for $n = \infty$ is due to Frabetti [8].

6. Steinberg–Leibniz superalgebras

In the preceding sections the Leibniz algebras that we considered are right Leibniz algebras in the sense of [25]. To extend these results to the Steinberg–Leibniz superalgebras, we need to describe them for left Leibniz algebras.

A left Leibniz algebra L is a vector space over a field K equipped with a K-bilinear map

$$[-,-]: L \times L \to L$$

satisfying the left Leibniz identity

$$\big[[a,b],c\big] = \big[a,[b,c]\big] - \big[b,[a,c]\big], \quad \forall a,b,c \in L.$$

In the following, Leibniz algebras always mean left Leibniz algebras.

For an associative dialgebra D, if we define a bracket by $[a,b] = a \vdash b - b \dashv a$, then $(D,[-,-])$ becomes a left Leibniz algebra. In this case, the inner derivation ad_m is defined by $\mathrm{ad}_m(x) = [m,x]$.

The Steinberg–Leibniz algebra $\mathfrak{stl}(n,D)$ (left) is defined by the same generators as the right Steinberg–Leibniz algebra and the following relations:

$$v_{ij}(k_1 a + k_2 b) = k_1 v_{ij}(a) + k_2 v_{ij}(b), \quad \text{for } a,b \in D,\ k_1,k_2 \in K; \tag{6.1}$$

$$[v_{ij}(a), v_{kl}(b)] = 0, \quad \text{if } i \neq l \text{ and } j \neq k; \tag{6.2}$$

$$[v_{ij}(a), v_{kl}(b)] = v_{il}(a \vdash b), \quad \text{if } i \neq l \text{ and } j = k; \tag{6.3}$$

$$[v_{ij}(a), v_{kl}(b)] = -v_{kj}(b \dashv a), \quad \text{if } i = l \text{ and } j \neq k, \tag{6.4}$$

where $1 \leqslant i \neq j \leqslant n$, $a \in D$. Similarly we can define the special linear left Leibniz algebra $\mathfrak{sl}(n,D)$.

It is clear that

$$[v_{ij}(a \vdash b), v_{kl}(c)] = [v_{ij}(a \dashv b), v_{kl}(c)]. \tag{6.5}$$

To describe results for left Leibniz algebras, we first introduce the modified Hochschild homology boundary d'_n, which is the K-linear map $D^{\otimes(n+1)} \to D^{\otimes n}$ given by the formula:

$$\begin{aligned} &d'_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \\ &\quad = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes (a_i \vdash a_{i+1}) \otimes \cdots \otimes a_n - a_1 \otimes \cdots \otimes a_{n-1} \otimes (a_n \dashv a_0). \end{aligned}$$

It is easy to check that $d'_n \circ d'_{n+1} = 0$ (see [17]). Thus we consider the nth homology group $HH'_n(D) = \mathrm{Ker}\, d'_n / \mathrm{Im}\, d'_{n+1}$.

Let I' be ideal of D generated by $(a \dashv b) \otimes c - (a \vdash b) \otimes c$, $\forall a, b, c \in D$. Clearly $I' \subset \operatorname{Ker} d_1$. Set $C_2'(D) = D \otimes D/(\operatorname{Im} d_2 + I')$ and $HHL_1(D) = \operatorname{Ker} d_1'/(\operatorname{Im} d_2' + I')$; then we also have the following exact sequence:

$$0 \to HHL_1(D) \to C_2'(D) \xrightarrow{d_1'} D. \tag{6.6}$$

Now Corollary 4.3 and Theorem 5.1 can be written as follows.

Theorem 6.1. *If $n \geqslant 3$ with $\operatorname{char} K \neq 3$ if $n = 3$ and $\operatorname{char} K \neq 2$ if $n = 4$, then the universal central extension of $\mathfrak{sl}(n, D)$ (left Leibniz algebra) is the Steinberg–Leibniz algebra $\mathfrak{stl}(n, D)$ (left) with kernel $HHL_1(D)$.*

Now we shall extend Theorem 6.1 to the Steinberg–Leibniz superalgebra $\mathfrak{stl}(m, n, D)$ for $n + m \geqslant 3$. First, we recall some notions about Leibniz superalgebras [16].

A Leibniz superalgebra [6] is a $\mathbb{Z}_2$-graded vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$ over a field K equipped with a K-bilinear map $[-,-] : L \times L \to L$ satisfying

$$[L_\sigma, L_{\sigma'}] \subset L_{\sigma+\sigma'}, \quad \forall \sigma, \sigma' \in \mathbb{Z}_2,$$

and the Leibniz identity

$$\big[[a,b],c\big] = \big[a,[b,c]\big] - (-1)^{|a||b|}\big[b,[a,c]\big], \quad \forall a, b, c \in L. \tag{6.7}$$

Obviously, $L_{\bar{0}}$ is a Leibniz algebra. Moreover any Lie superalgebra is a Leibniz superalgebra and any Leibniz algebra is a trivial Leibniz superalgebra. A Leibniz superalgebra is a Lie superalgebra if and only if

$$[a,b] + (-1)^{|a||b|}[b,a] = 0, \quad \forall a, b \in L.$$

The following Leibniz superalgebras were first introduced in [16].

The special linear Leibniz algebra $\mathfrak{sl}(m, n, D)$ has generators $E_{ij}(a)$, $1 \leqslant i \neq j \leqslant m + n$, $a \in D$, which satisfy the following relations:

$$[E_{ij}(a), E_{kl}(b)] = \delta_{jk} E_{il}(a \vdash b) - (-1)^{\tau_{ij}\tau_{kl}} \delta_{il} E_{kj}(b \dashv a). \tag{6.8}$$

Let D be a unital associative dialgebra, by definition, the Steinberg–Leibniz superalgebra $\mathfrak{stl}(m, n, D)$ is a Leibniz superalgebra generated by symbols $u_{ij}(a)$, $1 \leqslant i \neq j \leqslant m+n$, $a \in D$, subject to the relations

$$u_{ij}(k_1 a + k_2 b) = k_1 u_{ij}(a) + k_2 u_{ij}(b), \quad \text{for } a, b \in D,\ k_1, k_2 \in K; \tag{6.9}$$

$$[u_{ij}(a), u_{kl}(b)] = 0, \quad \text{if } i \neq l \text{ and } j \neq k; \tag{6.10}$$

$$[u_{ij}(a), u_{kl}(b)] = u_{il}(a \vdash b), \quad \text{if } i \neq l \text{ and } j = k; \tag{6.11}$$

$$[u_{ij}(a), u_{kl}(b)] = -(-1)^{\tau_{ij}\tau_{kl}} u_{kj}(b \dashv a), \quad \text{if } i = l \text{ and } j \neq k, \tag{6.12}$$

where $1 \leqslant i \neq j \leqslant m+n$, $a \in D$. It is clear that the relations (14)–(15) make sense only if $m+n \geqslant 3$. Moreover if $n+m \geqslant 3$, $\mathfrak{stl}(m,n,D)$ is perfect.

Using the same methods as in Sections 4 and 5 and [17], we can extend Theorem 6.1 to the Steinberg–Leibniz superalgebra $\mathfrak{stl}(n,D)$ for a unital associative dialgebra D for $m+n \geqslant 5$. The only differences are the definition of $h(a,b)$ and some corresponding relations. Here $h(a,b) = H_{1j}(a,b) - H_{1j}(1, b \dashv a)$.

Theorem 6.2. *Let D be a unital associative dialgebra and $m+n \geqslant 3$ with* $\operatorname{char} K \neq 3$ *if $m+n=3$ and* $\operatorname{char} K \neq 2$ *if $m+n=4$, then the universal central extension of the Leibniz superalgebra $\mathfrak{sl}(m,n,D)$ is $\mathfrak{stl}(m,n,D)$ with kernel $HHL_1(D)$, where $HHL_1(D)$ is defined as above* (*compare it with* [17]).

Remarks. 1. In [18], we study a special Leibniz algebra, the *Steinberg unitary Leibniz algebra*, and obtain some important results as in the Steinberg–Leibniz algebra case.

2. The above researches play key roles in studying the Leibniz algebras (superalgebras) graded by finite root systems. In fact, we proved the following results in [15,19].

Theorem 6.3 (Recognition Theorem of type A [15]). *Let L be a Leibniz algebra over K* ($\operatorname{char} K = 0$) *graded by the root system Δ of type A_l* ($l \geqslant 2$).

(1) *If $l \geqslant 3$, then there exists a unital associative K-dialgebra R such that L is centrally isogenous with $\mathfrak{sl}(n,R)$.*
(2) *If $l = 2$, then there exists a unital alternative K-dialgebra R such that L is centrally isogenous with $\mathfrak{stl}(n,R)$.*

Theorem 6.4 (Recognition Theorem of type $A(m,n)$ [19]). *Let L be a Leibniz superalgebra over K* ($\operatorname{char} K = 0$) *graded by the root system Δ of type $A(m,n)$* ($m > n \geqslant 0$). *Then there exists a unital associative superdialgebra R such that L is centrally isogenous with $\mathfrak{sl}(m+1, n+1, R)$.*

3. This research is also connected with ‘Leibniz K-theory,’ which was conjectured by J.L. Loday in [24].

There is an very interesting problem. What is the object ‘?’ in the following diagram?

$$\begin{tikzcd}
\textbf{Steinberg–Leibniz algebras} \arrow[r, "-"] \arrow[d] & \textbf{Steinberg–Lie algebras} \arrow[d] \\
? \arrow[r, "-"] & \textbf{Steinberg groups}
\end{tikzcd}$$

This structure will play a key role in studying the ‘Leibniz K-theory.’

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